647510 **MSC4MatC401x** **Seat No :________**

M.Sc.(Maths.) Semester - 4 (*CBCS*) Examination

March/April - 2024 (NEW COURSE)

Linear Algebra (Core)

Time: 02:30 Hours **Marks: 70**

Instructions:

1. All questions are compulsory.
2. Figures to the right indicate marks.

Q.1.(A) Answer any two questions out of three. **(10)**

(1) Let $\mathcal{A}$ be an algebra over F with unit element . Prove that $\mathcal{A}$ is isomorphic to a subalgebra of $A(V)$ for some vector space V over F.

(2) Let V b a finite dimensional vector space over F. Prove that $T \in A(V)$ is invertible if and only if the constant term in the minimal polynomial for T is not 0.

(3) Let V be an n-dimensional vector space over F and $T \in A(V)$. Suppose T has the matrix $m_1(T)$ and $m_2(T)$ with respect to basic $B_1 = \{v_1, v_2, \dots, v_n\}$ and $B_2 = \{w_1, w_2, \dots, w_n\}$ of V over F, respectively. Show that $m_1(T)$ and $m_2(T)$ are similar.

Q.1.(B) Answer any one question out of two. **(04)**

(1) Let V be a finite dimensional vector space over F. Show that $T \in A(V)$ is regular if and only if T maps V onto V.

(2) Let V be a finite dimensional vector space over F. Let $S, T \in A(V)$ and let S be regular. Prove that

(a) T and $S^{-1}TS$ have the same minimal polynomial.

(b) $\lambda \in F$ is a characteristic root of T if and only if it is a characteristic root of $S^{-1}TS$.

Q.2.(A) Answer any two questions out of three. **(10)**

(1) Let V be a finite dimensional vector space over F and $T \in A(V)$. If all the characteristic roots of T are in F, then show that there is a basis of V with respect to which the matrix of T is upper triangular.

(2) Prove that there exists a subspace W of V such that W is invariant under T and $V_1 \oplus W = V$.

(3) Let V be a finite dimensional vector space over F and $T \in A(V)$ be nilpotent with index of nilpotency n_1. Then prove that there is a basis of V in which the matrix of T is of the form $\begin{pmatrix} M_{n_1} & 0 & \cdots & 0 \\ 0 & M_{n_2} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & M_{n_n} \end{pmatrix}$

where $n_1 \geq n_2 \geq \cdots \geq n_k$ and $dimV = n_1 + n_2 + \cdots + n_k$.

Q.2.(B) Answer any one question out of two. **(04)**

(1) Let V be a finite dimensional vector space over F and $T \in A(V)$. If $\lambda \in F$ is a root of the minimal polynomial for T, then prove that λ is a characteristic root of T.

(2) Let V be a finite dimensional vector space over F. If $T \in A(V)$ has all its characteristic roots of T are in F, then prove that T satisfies a polynomial of degree n over F.

Q.3.(A) Answer any two questions out of three. **(10)**

(1) Let V be a finite dimensional vector space over F and $T \in A(V)$ be nilpotent. Then show that invariants of T are unique.

(2) Let V be a finite dimensional vector space over F and $T \in A(V)$. Let $V = V_1 \oplus V_2$, where V_1 and V_2 are subspaces of V invariant under T. Let $T_1 = T|v_1$ and $T_2 = T|v_2$. If the minimal polynomial of T_1 and T_2 over F are $p_1(x)$ and $p_2(x)$ then prove that the minimal polynomial of T over F is the least common multiple of $p_1(x)$ and $p_2(x)$.

(3) State and prove Jordan decomposition theorem.

Q.3.(B) Answer any one question out of two. **(04)**

(1) Find the invariants of the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (x_2, 0,0)$.

(2) Prove that two nilpotent linear transformation are similar if and only if they have the same invariants.

Q.4.(A) Answer any two questions out of three. **(10)**

(1) Let F be a field. Then for $A, B \in M_n(F)$ and $\lambda \in F$. Prove that

(a) $tr(\lambda A) = \lambda tr(A)$

(b) $tr(A + B) = tr(a) + tr(B)$

(c) $tr(AB) = tr(A)tr(B)$

(2) Let F be a field of characteristic 0, V be a vector space over F and $T \in A(V)$. Then prove that T is nilpotent if and only if $tr\left(T^i\right) = 0$, for all $i \geq 1$.

(3) For $A, B \in M_n(F)$ prove that $\det(AB) = \det(A) \det(B)$.

Q.4.(B) Answer any one question out of two. **(04)**

(1) State and prove Jacobson lemma.

(2) Prove that determinants of a matrix and its transpose are same.

Q.5.(A) Answer any two questions out of three. **(10)**

(1) Let V be an n-dimensional vector space over $\mathbb{R}$. Let f be a symmetric bilinear form on V which has rank r. Then prove that there is an ordered basis $\{\beta_1, \beta_2, \ldots, \beta_n\}$ for V in which the matrix of f is diagonal and such that $f\left(\beta_j, \beta_j\right) = \pm 1$; $j = 1, \ldots, r$. Furthermore, show that the number of basis vectors β_i for which $f\left(\beta_j, \beta_j\right) = 1$ is independent of the choice of basis.

(2) Let f be a bilinear form on the finite-dimensional vector space V. Let L_f and R_f be the linear transformations from V into V^* defined by $\left(L_f \alpha\right)(\beta) = f(\alpha, \beta) = (R_f \beta)\ (\alpha)$. Then prove that rank (L_f) =rank (R_f) .

(3) Let V be an n-dimensional vector space over $\mathbb{C}$. Let f be a non-degenerate symmetric bilinear form on V. Then the group preserving f is isomorphic to the complex orthogonal group $O(n, \mathbb{C})$.

Q.5.(B) Answer any one question out of two. **(04)**

(1) Let $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a map. Then prove that f is bilinear if and only if there exists $\alpha_{ij} \in \mathbb{R}$; $1 \leq i, j \leq n$ with $\alpha_{ij} = \alpha_{ji}$ such that $f(x, y) = \sum_{i,j=1}^{n} \alpha_{ij} x_i y_j$.

(2) Let V be a finite-dimensional vector space over a field of characteristic zero, and let f be a symmetric bilinear form on V. Then prove that there is an ordered basis for V in which f is represented by a diagonal matrix.

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